Section 16.2

Line Integrals

Scalar Line Integrals

Vector Line integrals

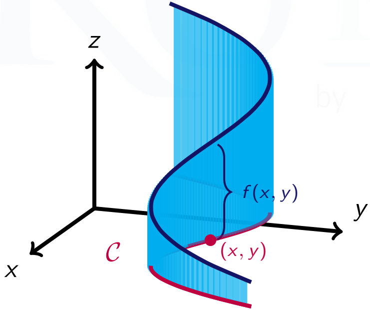
Work as a Vector Line Integral

1 Scalar Line Integrals

Scalar Line Integrals

Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^2$, and let $f(x, y)$ be a scalar-valued function.

The **line integral** $\displaystyle\int_{\mathcal{C}} f(x, y)\, ds$ is defined as the net area under the graph of f and over $\mathcal{C}$.



Link

- When $f(x, y) < 0$, the area under $\mathcal{C}$ and over the graph $f(x, y)$ contributes negatively to the integral.
- If $\mathcal{C} = [a, b]$ is a segment of the x-axis, then $\displaystyle\int_{\mathcal{C}} f(x, y)\, ds = \int_a^b f(x, 0)\, dx$.

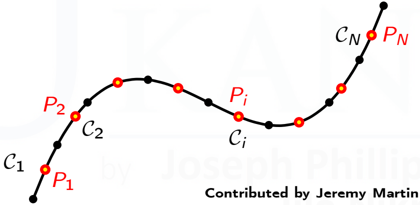
  Thus, single integrals are special cases of line integrals.

Scalar Line Integrals

To calculate the area under the surface $f(x,y)$ above a curve $\mathcal{C}$:

(i) Parametrize $\mathcal{C}$ by $\vec{r}(t)$ for $a \le t \le b$.

(ii) Subdivide $[a,b]$ into N subintervals of length Δt.
Let $P_i = \vec{r}(t_i^*)$ be a point in the subcurve $\mathcal{C}_i$ on $[t_{i-1}, t_i]$.



Contributed by Jeremy Martin.

(iii) The length of each subcurve is $\Delta s_i \approx \|\vec{r}\,'(t_i)\|\Delta t$. Let $N \to \infty$ to get the exact area:

$$\text{Area } = \int_{\mathcal{C}} f(x,y)\,ds = \lim_{N\to\infty} \sum_{i=1}^{N} f(P_i)\,\|\vec{r}\,'(t_i)\|\Delta t = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|dt$$

Scalar Line Integral Formula

If $\mathcal{C}$ is a smooth curve in $\mathbb{R}^2$ parametrized by a function $\vec{r}(t)$, and f is continuous on $\mathcal{C}$, then

$$\int_{\mathcal{C}} f(x,y)\,ds = \int_a^b f\left(\vec{r}(t)\right)\|\vec{r}\,'(t)\|\,dt.$$

- The same formula works for curves in $\mathbb{R}^n$ (for $n = 2, 3, \ldots$):

$$\int_{\mathcal{C}} f\;ds = \int_{\mathcal{C}} f(x_1, \ldots, x_n)\,ds = \int_a^b f\left(\vec{r}(t)\right)\|\vec{r}\,'(t)\|\,dt$$

- The symbol $ds = \|\vec{r}\,'(t)\|\,dt$ is called the **arc length element**. It represents a little bit of the arc length of the curve.

Scalar Line Integrals: Examples

Example 1: Evaluate $\int_{\mathcal{C}} 2 + x^2 y \, ds$, where $\mathcal{C}$ is the unit circle.

<u>Solution:</u>

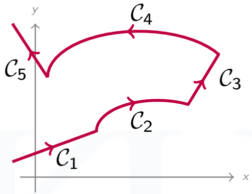
Step 1: Parametrize $\mathcal{C}$ by $\vec{\mathrm{r}}(t) = \langle \cos(t), \sin(t) \rangle$ for $t \in [0, 2\pi]$.

Step 2: Calculate $\|\vec{\mathrm{r}}'(t)\| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$.

Step 3: The arc length element is $ds = \|\vec{\mathrm{r}}'(t)\| \, dt = dt$, so

$$\int_{\mathcal{C}} 2 + xy^2 \, ds = \int_0^{2\pi} 2 + \cos^2(t) \, \sin(t) \, dt = 4\pi.$$

Piecewise-Smooth Curves



$\mathcal{C}$ is **piecewise-smooth** if $\mathcal{C}$ is the union of a finite number of smooth curves $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$. In that case,

$$\int_{\mathcal{C}} f\, ds = \int_{\mathcal{C}_1} f\, ds + \int_{\mathcal{C}_2} f\, ds + \cdots + \int_{\mathcal{C}_n} f\, ds$$

Example 2: Let $\mathcal{C}$ consist of the arc $\mathcal{C}_1$ of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segment $\mathcal{C}_2$ from $(1,1)$ to $(1,2)$. Evaluate $\displaystyle\int_{\mathcal{C}} 2x\, ds$.

$\mathcal{C}_1$: $\vec{r}_1(t) = \langle t, t^2 \rangle, \ 0 \le t \le 1$
$\|\vec{r}_1{}'(t)\| = \|\langle 1, 2t\rangle\| = \sqrt{1+4t^2}$

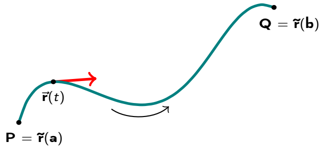
$\mathcal{C}_2$: $\vec{r}_2(t) = \langle 1, 1+t \rangle, \ 0 \le t \le 1$
$\|\vec{r}_2{}'(t)\| = \|\langle 0, 1\rangle\| = 1$

$$\int_{\mathcal{C}} 2x\, ds = \int_{\mathcal{C}_1} 2x\, ds + \int_{\mathcal{C}_2} 2x\, ds = \underbrace{\int_0^1 2t \overbrace{\sqrt{1+4t^2}}^{u=1+4t^2,\, du=8t\,dt}\, dt}_{\int_1^5 \frac{\sqrt{u}}{4}\, du = \frac{u^{3/2}}{6}\Big|_1^5} + \underbrace{\int_0^1 2\, dt}_{2} = \frac{5\sqrt{5}+11}{6}$$

2 Vector Line integrals

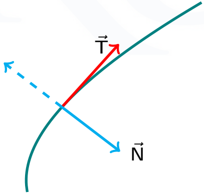
An **orientation** of a curve $\mathcal{C}$ is a choice of direction along the curve.

("Curve" = I-70; "oriented curve" = I-70 westbound)



The **unit tangent vector** to $\mathcal{C}$ points in the direction of motion of the parametrization.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

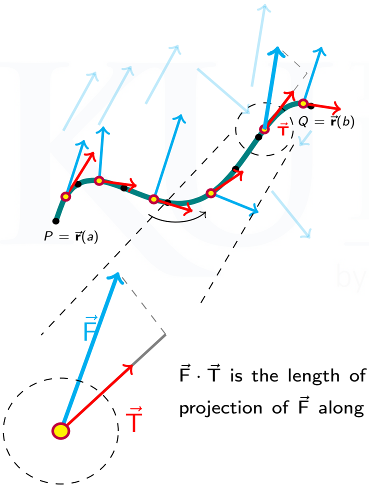


The **unit normal vector** to $\mathcal{C}$ is orthogonal to $\vec{T}$:

$$\vec{n}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Note: $\underbrace{\frac{d}{dt}(\overbrace{\|T\|^2}^{=1})}_{=0} = \frac{d}{dt}(T(t)\cdot T(t)) = \underbrace{2T(t)\cdot T'(t)}_{T(t)\cdot T'(t)+T'(t)\cdot T(t)}$ **So** $T(t) \perp T'(t)$.

Vector Line Integrals



$\vec{\mathsf{F}} \cdot \vec{\mathsf{T}}$ is the length of the projection of $\vec{\mathsf{F}}$ along $\vec{\mathsf{T}}$

The **tangential component** of a vector field $\vec{\mathsf{F}}$ at a point P on a curve $\mathcal{C}$ is the part of $\vec{\mathsf{F}}$ in the direction of the unit tangent vector:

$$\vec{\mathsf{F}}(P) \cdot \vec{\mathsf{T}}(P) = \|\vec{\mathsf{F}}(P)\| \cos(\theta)$$

We can measure "how much $\vec{\mathsf{F}}$ pushes an object moving along $\mathcal{C}$" by the integral of the tangential component:

$$\int_{\mathcal{C}} (\vec{\mathsf{F}} \cdot \vec{\mathsf{T}})\, ds.$$

This (scalar) quantity is the **vector line integral** of $\vec{\mathsf{F}}$ along $\mathcal{C}$.

Vector Line Integrals

The **vector line integral** of a vector field $\vec{F}$ over an oriented curve $\mathcal{C}$ is

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} (\vec{F} \cdot \vec{T})\, ds.$$

To compute the integral, let $\vec{r}$ be a parameterization, so that

$$(\vec{F} \cdot \vec{T})\, ds = \left(\vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \right) \|\vec{r}'(t)\|\, dt = \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

Vector Line Integral Formula

Let $\mathcal{C}$ be an oriented curve with a parametrization $\vec{r}(t)$ for $a \leq t \leq b$. Assume that $\vec{r}$ is **regular**, i.e., $\vec{r}'(t) \neq 0$ for all $t \in [a, b]$. Then

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} \;=\; \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$$

The Effects of Reversing Orientation

Let curve $\mathcal{C}$ be parametrized both by $\vec{r}(t)$, $a \le t \le b$, and $\vec{q}(u)$, $c \le u \le d$.

1. **Scalar line integrals are the same**, no matter whether $\vec{r}$ and $\vec{q}$ have the same or opposite orientations:

$$\int_{\mathcal{C}} f \, ds \;=\; \int_a^b f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt \;=\; \int_c^d f(\vec{q}(u)) \, \|\vec{q}\,'(u)\| \, du.$$

(Principle: The area of a wall is the same on both sides!)

2. **Vector line integrals depend on orientation.**

Same: $$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} \;=\; \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt \;=\; \int_c^d \vec{F}(\vec{q}(u)) \cdot \vec{q}\,'(u) \, du$$

Opposite: $$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} \;=\; \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt \;=\; -\int_c^d \vec{F}(\vec{q}(u)) \cdot \vec{q}\,'(u) \, du$$

Example 3: Let $\vec{F}(x,y,z) = \left\langle x^2,\ y^2,\ yz \right\rangle$ and let $\mathcal{C}$ be parametrized by $\vec{r}(t) = \langle \cos(t),\ \sin(t),\ t \rangle$ on $[0,\pi]$. Evaluate $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$.

Solution:

$$\begin{aligned}
\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} &= \int_0^{\pi} \vec{F}\left(\vec{r}(t)\right) \cdot \vec{r}\,'(t)\,dt \\
&= \int_0^{\pi} \left\langle \cos^2(t),\ \sin^2(t),\ \sin(t)t \right\rangle \cdot \langle -\sin(t),\ \cos(t),\ 1 \rangle\ dt \\
&= \int_0^{\pi} \underbrace{-\sin(t)\cos^2(t)dt}_{u=\cos(t),\ du=-\sin(t)dt} + \int_0^{\pi} \underbrace{\sin^2(t)\cos(t)dt}_{u=\sin(t),\ du=\cos(t)dt} + \int_0^{\pi} \underbrace{t\sin(t)\,dt}_{u=t,\ dv=\sin(t)dt} \\
&= \frac{\cos^3(t)}{3} + \frac{\sin^3(t)}{3} - t\cos(t) + \sin(t) \Big]_0^{\pi} = \pi - \frac{2}{3}
\end{aligned}$$

Notation for Line Integrals

Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ be a vector field and $\mathcal{C}$ a curve parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $[a,b]$. There are many different ways to write the line integral of $\vec{F}$ over $\mathcal{C}$:

- Vector Differential Form: $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds$
- Parametric Vector Evaluation: $\displaystyle\int_a^b \vec{F}\big(x(t), y(t), z(t)\big) \cdot \vec{r}\,'(t)\, dt$
- Parametric Scalar Evaluation:
  $$\int_a^b \Big(P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t))y'(t) + R(x(t),y(t),z(t))z'(t) \Big)\, dt$$
- Scalar Differential Form: $\displaystyle\int_{\mathcal{C}} P\, dx + Q\, dy + R\, dz$

3 Work as a Vector Line Integral

Work as a Line Integral

Let $\mathcal{C}$ be a smooth curve parametrized by $\vec{r}(t)$ on $[a, b]$. Let $\vec{F}$ be a continuous vector field over a region containing $\mathcal{C}$.

If $\vec{F}$ measures force, then the **work** done on an object that moves from $\vec{r}(a)$ to $\vec{r}(b)$ along $\mathcal{C}$ is

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}.$$

- Work measures the energy expended by the field in moving the object,
- Work can be negative (which indicates that the object expends energy in moving against the field).
- If $\vec{F}$ is constant, then work is just $\vec{F} \cdot \overrightarrow{PQ}$, where $P = \vec{r}(a)$ and $Q = \vec{r}(b)$.

Flux Across a Plane Curve (Optional)

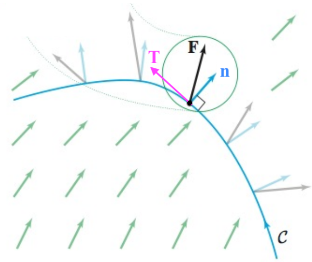
Work is the integral of force **along** a curve $\mathcal{C}$ (in the tangent direction).

What about the integral of $\vec{\mathsf{F}}$ **across** $\mathcal{C}$ (i.e., in the normal direction)?

Suppose $\mathcal{C}$ is parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$. Then

$$\vec{n}(t) = \frac{\overbrace{\langle y'(t),\ -x'(t)\rangle}^{\vec{N}}}{\|\vec{r}\,'(t)\|}$$

is a unit normal vector to $\mathcal{C}$ at $\vec{r}(t)$. (So is $-\vec{n}(t)$.)



The **flux** across $\mathcal{C}$ is the line integral of the normal component $\vec{F} \cdot \vec{n}$:

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{n}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{N}(t)\, dt$$

Flux measures flow of the field across $\mathcal{C}$ in the direction of $\vec{n}$.